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INTRODUCTION

Entanglement entropy in extended quantum systems

The quantum description of nature is very peculiar. In the conventional understanding, it asserts that there is a fundamental uncertainty in what we measure which we can never overcome; it also correctly predicts a wealth of phenomena that seem to disagree with classical mechanics, like interference of particles and quantisation of energy levels. However, it is arguably the phenomenon of *entanglement* that is the most fundamental and potentially disturbing characteristic distinguishing the quantum from the classical worlds. It was one of the first aspects of quantum physics to be studied and discussed (see the classics by Einstein, Podolsky and Rosen [1] and by Schrödinger [2], both dating back to 1935), and there are many ways of defining it. We could describe it as follows: it implies that the measurement of an observable of a subsystem may affect drastically and *instantaneously* (according to the conventional Copenhagen interpretation) the possible outcome of a measurement on another part of the system, no matter how far apart it is spatially. This is to be distinguished from the phenomenon of classical correlation, where the distribution of the possible outcomes of a measurement on one part of the system may depend on the outcome of a previous measurement elsewhere, but is strictly limited by the speed of propagation of signals.

Quantum entanglement apparently leads to ‘spooky’ connections between subsystems that may be arbitrarily far apart in space. It is entanglement that forbids an explanation of the quantum randomness via hidden variables (Bell’s inequalities [3]), that allows some quantum algorithms to be much more efficient than their best classical counterparts (e.g. Shor’s algorithm [4]), and that allows the possibility of quantum teleportation.

In the last 30 years, interest in quantum entanglement has risen sharply in various formerly disconnected scientific communities, bringing them together in unexpected ways. In the early 1980s, the entanglement between quantum states with support both inside and outside of a black hole, arising for instance from particle pair creation near the event horizon, was suggested to be the basis for the properties of Hawking’s radiation, in particular for the associated Beckenstein–Hawking entropy. Technically, the idea is that in a pure, bi-partite state, an observer who can only measure one subsystem (e.g. outside the black hole) will perceive an effective *mixed* quantum state if there is entanglement with the rest of the system (e.g. inside the black hole). The corresponding entropy is the von Neumann entropy associated with the reduced density matrix – this is the *entanglement entropy* of a quantum subsystem. Although this idea does not provide the full explanation, it is nevertheless true that, like the Beckenstein–Hawking entropy, entanglement entropy, in many situations, grows like the area of the region separating the subsystems (in fact, it is certain quantum corrections to the black hole entropy that are given by the entanglement entropy). The idea that entanglement between subsystems of a pure state gives rise to effective mixed states is also used in the decoherence theory of quantum measurements.

Later on, in the 1990s, the necessity of providing a quantitative measure of entanglement was understood in the science of quantum information, since, in this context, entanglement is an important *resource*. Although it is rather straightforward to determine whether entanglement between two subsystems exists, how do we *quantify* it? There are in fact many measures of entanglement that find applications in different situations. But an important principle is that of an ordering with respect to operations that 'naturally' cannot increase entanglement. Local unitary operations (unitary transformations completely lying in the subspace corresponding to one subsystem) are not expected to change entanglement between subsystems, and classical communications are expected at best to decrease it. Hence, if the fact that a state can be obtained from another via local operations and classical communication (LOCC) provides an ordering in the space of states, then any real function of these states respecting this will give rise to a useful entanglement measure. One of the most important results in quantum information science [5] is that there is indeed such an ordering in the space of pure states, and that the *entanglement entropy* (or any monotonic function of it) is a candidate for the corresponding measure: it cannot increase under LOCC. For mixed states, the situation is more complicated, but a recent proposed solution to the problem can be found in [6].

The study of black holes naturally led to the more general consideration of entanglement entropy in quantum field theory (QFT), and that of quantum information to the corresponding measure in spin chains. Since these two subjects naturally overlap in the context of many-body systems (in particular close to criticality, see below), it soon became apparent that there was a rich structure to be uncovered by studying entanglement entropy in this context, and by putting together ideas from field theory and quantum information science. This is the subject of the present volume, which groups together reviews by leading experts in the field of entanglement entropy and related notions in many-body, extended, quantum systems.

Many-body problems are notoriously difficult to handle. In extended quantum systems, a macroscopically large number of degrees of freedom interact in a *local* way. For our purposes, this means that the degrees of freedom can be seen as lying in $\mathbb{R}^d$, in such a way that the whole system is extended over distances much larger than the interaction range (the *dimensionality* of the system is the lowest possible d where this is possible). Locality helps in many ways in understanding such systems. For instance, when they present *critical behaviour*, where correlation lengths become large, the powerful techniques of QFT give access to universal properties in the neighbourhoods of critical points. Ideas of entanglement then provide a fresh point of view on extended quantum systems and QFT.

The reasons for the success entanglement entropy study are numerous. Its simple definition, compared to other entanglement measures, makes it amenable to study in extended quantum systems, in particular via the 'replica trick'. Also, the entanglement entropy displays universal behaviour near critical points, and provides a good characterisation of universal aspects of quantum states. It extracts fundamental properties of critical neighbourhoods in a 'cleaner' way than most standard quantities, like quantum correlation functions: the central charge (without the need for knowing *a priori* the speed of sound), the mass spectrum (in a functional form independent of the scattering matrix), the boundary entropy, the topological charge, and so on. Moreover, it is easy to argue that it measures quantum correlations in a more universal, canonical and flexible way than do correlation functions themselves. Indeed, there is no need for the precise characterisation of any local observable; and we only have to provide a partition of a complete set of compatible observables in order to determine the division into subsystems and calculate the entanglement entropy. This means that we don't have to know the properties (like scaling dimensions) and correlations of any particular local observable; and that we can in fact choose to partition the Hilbert space according to different observables than local ones (for instance, particle partitioning).

Entanglement entropy has a very natural definition. Consider an arbitrary quantum system prepared in a pure state $|\Psi\rangle$ (assumed to be normalised to unity), so that it has the density matrix $\rho = |\Psi\rangle\langle\Psi|$. We suppose that the Hilbert space can be written as a tensor product $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$. We imagine two observers (traditionally named Alice and Bob) such that Alice can make observations only in $\mathcal{H}_A$ (corresponding to observables of the form $\mathcal{O}_A \otimes 1_B$), and correspondingly for Bob. In general, Alice's observations are entangled with those of Bob. One of the most useful mathematical tools in understanding how to quantify entanglement is that of the Schmidt decomposition, which is based on the property of singular value decomposition for matrices. It states that any pure state $|\Psi\rangle$ may be written as

$$|\Psi\rangle = \sum_j c_j |\psi_j\rangle_A \otimes |\psi_j\rangle_B ,$$

where $|\psi_j\rangle_{A,B}$ are orthonormal vectors in $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively (note that there is only one sum here: for each vector in $\mathcal{H}_A$ there is just one vector in $\mathcal{H}_B$). If all the states are normalised to unity, $\sum_j |c_j|^2 = 1$. Moreover, the c_js can be chosen to be real and $\geqslant 0$. The measure of the entanglement between A and B in $|\Psi\rangle$ that we then consider is the entropy

$$S \equiv -\sum_j |c_j|^2 \log |c_j|^2 .$$

If $c_1 = 1$ and all the rest vanish, $|\Psi\rangle$ is a product state and is unentangled (although there may still be correlations). If, on the other hand, all the c_js are equal, then S takes its maximal value, given by the logarithm of the smaller of the dimensions of $\mathcal{H}_A$ and $\mathcal{H}_B$. For example, if each subspace is a tensor product of N qubits (spin-$\frac{1}{2}$ degrees of freedom) then the maximal entanglement entropy is $N \log 2$. (We note that in the quantum information literature it is customary to take all logarithms to base 2. However, the use of the replica trick naturally leads to base e.)

Equivalently, we can define the entanglement entropy as the von Neumann entropy

$$S_A = -\mathrm{Tr}_{\mathcal{H}_A} \rho_A \log \rho_A$$

of Alice's reduced density matrix

$$\rho_A = \mathrm{Tr}_{\mathcal{H}_B} \rho.$$

Clearly, $S_A = S_B = S$ when ρ corresponds to a pure state, while in general $S_A \neq S_B$ for a mixed one. This definition makes it obvious that the entanglement entropy is basis independent: it is a fundamental universal feature. Another property that makes S_A interesting in extended systems is *sub-additivity*: if the Hilbert space is the tensor product of a set of 'local' degrees of freedom, then given two subsets A_1 and A_2 (where $A_1 \cup A_2$ is *not* necessarily the whole set), the entanglement entropy in any pure state satisfies the inequality

$$S_{A_1} + S_{A_2} \geqslant S_{A_1 \cup A_2} + S_{A_1 \cap A_2} .$$

There is a completely elementary way of understanding basic properties of reduced density matrices for pure states, using the anti-linear maps $f : \mathcal{H}_A \to \mathcal{H}_B$ given by $f|\psi\rangle_A = {}_A\langle\psi|\Psi\rangle$, and $\tilde{f} : \mathcal{H}_B \to \mathcal{H}_A$ given by $\tilde{f}|\psi\rangle_B = {}_B\langle\psi|\Psi\rangle$ (the Schmidt decomposition is essentially the singular value decomposition for such maps). With these, we can write $\rho_A = \tilde{f} f$ and $\rho_B = f \tilde{f}$. This makes it evident, for instance, that ρ_A and ρ_B have the same set of non-zero eigenvalues, with the same degeneracies, giving in particular $S_A = S_B$.

Apart from the enormous theoretical interest in understanding of the entanglement in extended systems, in the context of quantum field theory, entanglement entropy turns out to have a deep geometrical meaning. Through the replica trick, which stems from the simple identity

$$-\mathrm{Tr}_{\mathcal{H}_A} \rho_A \log \rho_A = -\left(\frac{d}{dn} \mathrm{Tr}_{\mathcal{H}_A} \rho_A^n\right)_{n=1} ,$$

the entanglement entropy is related to partition functions of the n-independent-copy model orbifolded by elements of the $\mathbb{Z}_n$ group. In 1+1 dimensions, these are partition functions on multi-sheeted presentations of Riemann surfaces. Such partition functions, for instance, have connections to τ-functions (isomonodromic or from integable hierarchies) both in conformal cases and in massive free models, although some work is still needed to completely clarify the situation. On a different vein, by the holographic principle, entanglement entropy can be associated to geometric objects in gravity theories, like areas of certain minimal surfaces in anti-de Sitter space. In this sense, it has helped in understanding further the holographic principle.

The study of entanglement entropy has also had a great impact on numerical algorithms and on their development. In fact, in any algorithm running on a classical computer the quantum features are encoded in so-called tensor states and in their connections. In matrix product states (MPSs) – that are the basis of the celebrated density matrix renormalization group – the amount of entanglement entropy that can be 'stored' in a matrix of dimension M scales like $\log M$. Thus, to have an accurate description of a given quantum state, the dimension of the tensor should be proportional to the exponential of the maximum entanglement entropy of any subsystem. In one dimension this implies that M scales at most linearly with the system sizes, but in higher dimensions, the area law requires an exponentially growing computational resource. This explains the success of the density matrix renormalization group in 1D and its failure in higher dimensions. However, this is not yet the end of the story. The understanding of entanglement in extended systems has allowed the design of new classes of tensor states (e.g. tree tensor states, multiscale entanglement renormalization Ansatz, and Projected Entangled Pair States, etc.), with a structure that is specifically organized in such a way as to store the desired amount of entanglement in a relatively small matrix, hence to require at most polynomial resources to store and manipulate quantum states on a classical computer.

This characterisation has also helped dramatically in understanding non-equilibrium situations. In fact, nowadays it is well understood that in generic non-equilibrium dynamics under quantum evolution, the entanglement entropy grows with time up to a maximum scaling with subsystem size (with important exceptions, such as local quantum quenches). This implies that no matter how small the perturbation is, the long-time evolution can be obtained only for relatively small subsystem sizes with MPSs. Conversely, the short-time dynamics is effectively described by MPSs. These features have boosted the research to find tensor states for non-equilibrium dynamics.

The goal of this issue is to present a self-contained introduction to most of the topics that gravitate around entanglement entropy in extended quantum systems, with the hope of being complementary to the already existing ones: the review by Amico, Fazio, Osterloh and Vedral [7], discussing the zero and finite temperature properties of bipartite and multipartite entanglement in interacting spin, fermionic and bosonic model systems; and the one by Eisert, Cramer and Plenio [8], considering the area law. Both these topics will only be marginally discussed here, referring the reader to the two above-mentioned reviews. This issue consists of three main blocks. The first four reviews are introductory to the subject, the second block of four describes the quantum field theory approach to entanglement, while the last four reviews consider four specific topics of large interest for condensed matter physics. In detail, the content of this issue is as follows:

- Amico and Fazio [9] open this issue with a review about the various measures of entanglement in extended systems, with particular emphasis on the connection with magnetic order and criticality.
- Latorre and Riera [10] introduce the concept of entanglement entropy by considering in

detail the simplest quantum spin systems, giving an overview of results and methods.

- Peschel and Eisler [11] review the properties of reduced density matrices for free fermionic or bosonic many-particle systems in their ground state and in situations out of equilibrium.
- Cirac and Verstraete [12] discuss the different descriptions of many-body quantum systems in terms of tensor product states and their applications to numerical algorithms.
- Calabrese and Cardy [13] review the conformal field theory approach to the entanglement entropy for the ground state of critical systems and for out-of-equilibrium situations.
- Castro-Alvaredo and Doyon [14] consider the case of massive one-dimensional quantum field theories within the form-factor approach for integrable models and more generally for massive quantum field theory.
- Casini and Huerta [15] introduce general methods to calculate the entanglement entropy for free fields, within the Euclidean and the real time formalisms in any dimensionality.
- Nishioka, Takayanagi, and Ryu [16] review recent progresses on the holographic understandings of entanglement entropy in the context of the AdS/CFT correspondence.
- Affleck, Laflorencie, and Sorensen [17] consider a number of situations where a quantum impurity or a physical boundary has an interesting effect on entanglement entropy.
- Refael and Moore [18] discuss the entanglement entropy in systems with quenched randomness, concentrating on universal behavior at strongly random quantum critical points.
- Fradkin [19] considers the case of topological order in two dimensions, especially in models with a conformal invariant wave function, having applications to quantum dimer models and fractional quantum Hall states.
- Finally, Haque, Zozulya, and Schoutens [20] consider a different bipartition of the quantum states, when the subsystem consist of a given number of itinerant particles and not of a spatial subset.

Many interesting topics connected with entanglement entropy in extended systems are unfortunately not present in this issue. We apologise for these omissions and for the personal choice of topics included, and realise that this cannot satisfy all scientists working in the field. However, we hope that this issue can serve as a useful introduction to newcomers in the field, and as a convenient and complete enough reference for the experts.

Pasquale Calabrese, *Dipartimento di Fisica dell'Universitá di Pisa and INFN, Italy*
John Cardy, *Rudolf Peierls Centre for Theoretical Physics, Oxford University and All Souls College, UK*
Benjamin Doyon, *Department of Mathematics, King's College London, UK*
(work done while at: Department of Mathematical Sciences, Durham University, UK)
Guest Editors

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